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A Twist on Chiral Potts[1]

Brian Davies[1]

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We show that the chiral Potts model may be formulated so that the rapidity lines carry a second integer variable—an increment or "twist" in each bond crossing it. This modification does not affect those properties of the chiral Potts model which lead to integrability, since it is equivalent to one of the automorphisms allowed for in the theory. In particular, transfer matrices still form commuting families and still satisfy hierarchies of functional equations. Surprisingly, the superintegrable case with twists retains the special algebraic properties which lead to its Ising-like spectra. The formalism should be useful for considering systems with twisted boundary conditions or embedded interfaces.

KEY WORDS: Boundary conditions; chiral Potts; exact solution; statistical mechanics.

1. INTRODUCTION

Recently much progress has been made in the solution of the chiral Potts model. This is a class of N-state two-dimensional lattice models for which solutions of the star-triangle relation[1] have been found. It is Z_N symmetric, but chirally asymmetric: that is, the Boltzmann weight for an adjoining pair of spins n, n' depends only on the difference $n-n'$, but is asymmetric under interchange of n, n' (mod N). The star-triangle property implies, as an immediate corollary, that there are commuting families of transfer matrices parametrized by some "rapidity" variables and also a "temperature-like" variable. Further, each commuting family generates an infinite sequence of conserved quantities, the simplest of which is an N-state spin chain Hamiltonian involving only nearest neighbor interactions. For a special choice of rapidity variables—the "superintegrable"

[1] Mathematics Department, Faculty of Science, Australian National University, Canberra, ACT 2601, Australia.

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case—there is an underlying Lie algebra identical with the algebra whereby Onsager solved the Ising model,[2] and this has led to many new results in this case.[3-7]

We recall some basic results for the chiral Potts model.[1,8] The parametrization employs algebraic curves satisfying the homogeneous equations

$$a_p^N + k' b_p^N = k d_p^N, \qquad k' a_p^N + b_p^N = k c_p^N, \qquad k^2 + k'^2 = 1 \tag{1}$$

There are two types of interaction between neighboring spins, each labeled by a pair of rapidities—see Fig. 1. The weights are defined by

$$\begin{aligned} W_{pq}(n)/W_{pq}(0) &= \prod_{k=1}^{n} (d_p b_q - a_p c_q \omega^k)/(b_p d_q - c_p a_q \omega^k) \\ \bar{W}_{pq}(n)/\bar{W}_{pq}(0) &= \prod_{k=1}^{n} (\omega a_p d_q - d_p a_q \omega^k)/(c_p b_q - b_p c_q \omega^k) \end{aligned} \tag{2}$$

where $\omega = \exp(-2\pi i/N)$. The convention of Fig. 1, including the chirality which is implicit in it, is most important in what follows. For $W_{pq}(n)$ the arrow on the bond points to the right of the rapidity p, while q points to the left of p. For $\bar{W}_{pq}(n)$ both arrows point to the left of p. Thus, there is no ambiguity in assigning spins and Boltzmann weights once the rapidity lines are given, and there will be no ambiguity in assigning "twists" to bonds. There are various automorphisms of the algebraic curves (1), notably

$$\begin{aligned} R&: \quad a_{Rq}, b_{Rq}, c_{Rq}, d_{Rq} = b_q, \omega a_q, d_q, c_q \\ T&: \quad a_{Tq}, b_{Tq}, c_{Tq}, d_{Tq} = \omega a_q, b_q, \omega c_q, d_q \\ U&: \quad a_{Uq}, b_{Uq}, c_{Uq}, d_{Uq} = \omega a_q, b_q, c_q, d_q \end{aligned} \tag{3}$$

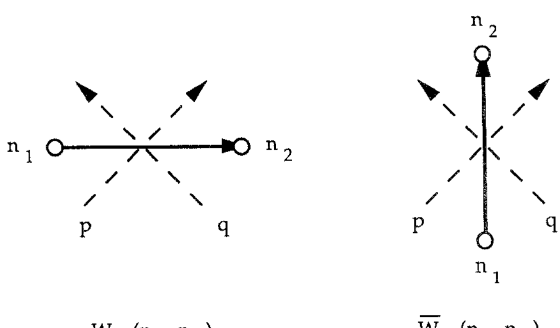

Fig. 1. Boltzmann weights for the chiral Potts model.

The weights satisfy the star-triangle relation, shown diagramatically in Fig. 2. We attach an extra variable, to be explained below, to each rapidity line, so that the relation reads

$$\sum_{n=1}^{N} W_{(pn_p)(rn_r)}(n_1-n)\, \bar{W}_{(qn_q)(rn_r)}(n_2-n)\, \bar{W}_{(pn_p)(qn_q)}(n-n_3)$$
$$= R_{pqr}\, \bar{W}_{(pn_p)(rn_r)}(n_2-n_3)\, W_{(qn_q)(rn_r)}(n_1-n_3)\, W_{(pn_p)(qn_q)}(n_1-n_2) \quad (4)$$

A second form results from negating all arrows (or equivalently, all spins). The difference between the two is in the "net circulation" of rapidity about the star point. We construct transfer matrices for transfer in the diagonal direction as shown in Fig. 3. This follows the convention of Baxter *et al.*,[9] a paper to which we shall refer as FR. We shall refer to equations from Baxter *et al.* by attaching the prefix FR to an equation number. The lattice is wound on a cylinder, so that the last site in each row of L sites is adjacent to the first. The standard use of the star-triangle relation is as follows: Regarded as $N \times N$ matrices, the Boltzmann weights $W_{qr}(n-n')$ are invertible, hence we may insert a suitably chosen matrix multiplied by its inverse into the product of two transfer matrices to produce a star point. On using the star-triangle relation $2L$ times, we find that transfer matrices satisfy the commutation relation that $T_q \hat{T}_r$ is a scalar multiple of $T_r \hat{T}_q$ provided only that the vertical rapidities all match. This is Eq. (FR2.32b). The product $T_q \hat{T}_r$ is a "checkerboard" transfer matrix: these form two-parameter commuting families. The commutation relation $(T_q \hat{T}_r)(T_{q'} \hat{T}_{r'}) = (T_{q'} \hat{T}_{r'})(T_q \hat{T}_r)$ [Eq. (FR2.33)] for this checkerboard model may be demonstrated diagramatically by repeated use of the star-triangle relation exactly as in Fig. 9 of ref. 10 for the checkerboard Ising model. Because the star-triangle and triangle-star transformations are paired, there is exact commutation. Each commuting family is labeled by the values of k', q, r,

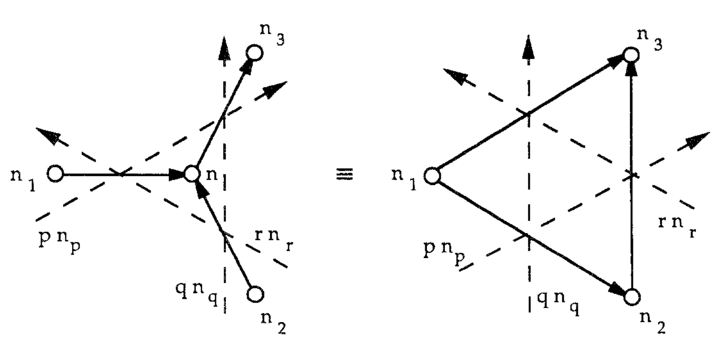

Fig. 2. Star-triangle relation for the chiral Potts model.

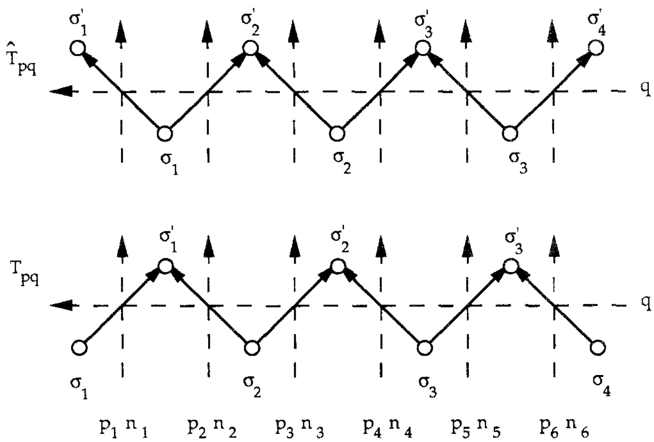

Fig. 3. The two transfer matrices T and $\hat{T}$.

and the $2L$ vertical rapidities. Moreover, the Hermitian adjoint of a matrix $(T_q \hat{T}_r)$ is in the same commuting family as $(T_q \hat{T}_r)$ because of the properties of the weights, so they are normal (diagonable) matrices.

Functional relations play a crucial part in the analysis of the chiral Potts model.[9,11–13] The derivations of FR involve some very long computations, and are very general indeed, since they apply to row transfer matrices in which the vertical rapidities may be chosen independently—the general "inhomogeneous" case. Initial interest in the physical properties of a solvable model is usually for the homogeneous case with periodic boundary conditions, and this is already very complicated for the chiral Potts model. For finite-size calculations, and also calculations of surface properties, one may be interested in the homogeneous case with one or more columns of dislocations.[14–18] A typical dislocation would be a "twist" in the boundary condition, identifying the spin variable at site L with that at site 1 *plus* an integer twist. The purpose of this paper is to point out that such generalizations are already contained in FR, and to explicitly exhibit the effect of twisting the bonds, particularly the effect on the "tau" matrices. Specifically, we see that chiral Potts transfer matrices form commuting families even when arbitrary increments in the spin variable are associated with the rapidities, and that the functional relations for transfer matrices of the chiral Potts model include this general case. These relations are of great importance for calculating eigenvalues and other properties of the model.[12,13]

Commuting families of transfer matrices generate Z_N spin-chain Hamiltonians of the form $H = A_0 + kA_1$, with which they commute. In the homogeneous superintegrable case with periodic boundary conditions, it is known that the two-component operators A_0, A_1 of the Hamiltonian satisfy the Dolan–Grady condition[19] and so generate an Onsager algebra.[7] This is a Lie algebra whose semisimple part is the direct sum of the algebra $su(2)$, which accounts for the Ising-like property—the "superintegrability." In the inhomogeneous case this property is certainly not true, even for the Ising model. However, we shall show that A_0, A_1 satisfy the Dolan–Grady condition with arbitrary twists provided that the rapidities are otherwise homogeneous. This surprising result confirms some conjectures about relations between the spectra of these Z_N Hamiltonians in the various sectors made by von Gehlen and Rittenberg[20] on the basis of numerical calculations. It should also have further applications to the superintegrable case.

2. TWISTED BONDS

Referring to Figs. 1 and 2, attach a second (integer) variable n_p to each rapidity line p, which will become a twist in the bonds crossed by that line. Observe that the star-triangle relation (4) is a set of N^3 equations labeled by the integers n_1, n_2, n_3. They are invariant under the replacement $n_i \to n_i + n_p$. Such a replacement may be absorbed into the definition of the Boltzmann weights. Thus, we write (pn_p) in place of p, and define

$$W_{(pn_p)(qn_q)}(n) = W_{pq}(n + n_p + n_q), \qquad \bar{W}_{(pn_p)(qn_q)}(n) = \bar{W}_{pq}(n - n_p + n_q) \tag{5}$$

From Eqs. (2) and (3), we see that there is an equivalent definition which uses the automorphism T:

$$W_{(pn_p)(qn_q)}(n) = W_{T^{n_p}p,\, T^{n_q}q}(n), \qquad \bar{W}_{(pn_p)(qn_q)}(n) = \bar{W}_{T^{n_p}p,\, T^{n_q}q}(n) \tag{6}$$

However, the notation of Eq. (5) is chosen to try to maintain consistency with the conventions of FR while explicitly exhibiting the twists. In general, we label a rapidity line by the pair (p, n_p), with the convention that $(p, 0)$ is just written as p. Also, where the rapidities p_j are labeled by an integer, we write n_j for the corresponding twist variable. The integer attached to each rapidity line indicates the effect is has on the spin difference: this effect depends on whether the arrow joining the two spins crosses the rapidity from right to left or vice versa. It obviously makes no difference to the star-triangle relation or to the commutation of transfer matrices. Transfer matrices with different twist sets n_j, $j = 1,..., 2L$, are not equivalent under similarity transformation, as is evident already in the Ising case, where one antiferromagnetic bond changes the entire spectrum. There is no point in attaching a twist variable to a horizontal rapidity; this is equivalent to multiplying the transfer matrix by the spin shift operator [see Eq. (FR 2.43)].

The derivations of FR for the functional relations involve some very long computations. Most of them may be taken over directly here: the main difference is that it is essential to consider the general inhomogeneous case, with the extra factors introduced by the twist variables. The "first hierarchy" of functional relations is related to particular combinations of the automorphisms (3). Specifically, we denote by $q \to \bar{q}(k, l)$ the automorphism [Eq. (FR 2.37)]

$$a_{\bar{q}(k,l)},\ b_{\bar{q}(k,l)},\ c_{\bar{q}(k,l)},\ d_{\bar{q}(k,l)} = \omega^k b_q,\ \omega^l a_q,\ d_q,\ c_q \tag{7}$$

The checkerboard transfer matrix $T_q \hat{T}_r$ is built up using star weights shown in Fig. 4. We have

$$(T_q \hat{T}_r)_{\sigma\sigma'} = \prod_{j=1}^{L} U_{(j)qr}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) \tag{8}$$

where the label (j) is an abbreviated notation for the rapidity pair $(pn)_{2j-1}(pn)_{2j}$. Using the label j for $p_{2j-1}p_{2j}$, $U_{(j)qr}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j)$ is related to the star weight $U_{jqr}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j)$ of FR by

$$\begin{aligned} &U_{(j)qr}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) \\ &\quad = U_{jqr}(\sigma_j + n_{2j-1}, \sigma_{j+1} - n_{2j}, \sigma'_{j+1} - n_{2j}, \sigma'_j + n_{2j-1}) \end{aligned} \tag{9}$$

Now let $r = \bar{q}(k, l)$, where the integers k, l satisfy $1 \leqslant k + l \leqslant N$, $k \geqslant 0$, $l > 0$, and let ζ_{kl} denote the set of consecutive integers $\{-k, ..., l-1\}$. Then it is shown in FR that the star weights have the property

$$U_{(j)qr}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) = 0 \quad \text{if} \quad \sigma_j - \sigma'_j \in \zeta_{kl} \quad \text{and} \quad \sigma_{j+1} - \sigma'_{j+1} \notin \zeta_{kl} \tag{10}$$

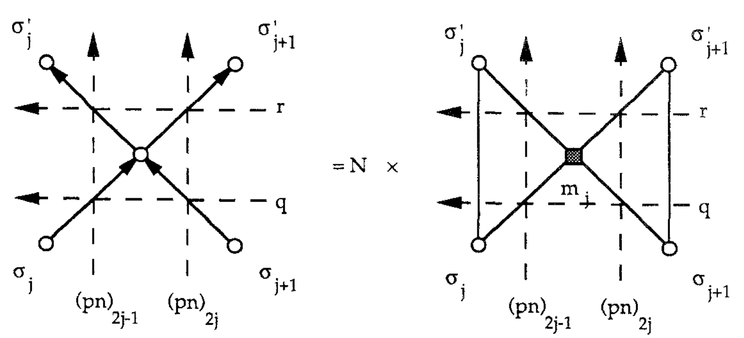

Fig. 4. Star weight of Eq. (9).

The first hierarchy stems from this fact, which implies the fundamental decomposition [Eq. (FR 3.18)] that $T_q \hat{T}_r$ is the sum of two matrices $\mathscr{T}_q^{(k,l)}$ and $\bar{\mathscr{T}}_q^{(k,l)}$ with

$$\begin{aligned}(\mathscr{T}_q^{(k,l)})_{\boldsymbol{\sigma\sigma}'} &= 0 \qquad \text{unless} \quad \sigma_j - \sigma'_j \in \zeta_{kl} \quad \forall j \\ (\bar{\mathscr{T}}_q^{(k,l)})_{\boldsymbol{\sigma\sigma}'} &= 0 \qquad \text{unless} \quad \sigma_j - \sigma'_j \notin \zeta_{kl} \quad \forall j\end{aligned} \tag{11}$$

The utility of this relation depends on calculating the entries of the matrices $\mathscr{T}_q^{(k,l)}$. As in FR, the matrices $\mathscr{T}_q^{(k,l)}$ and $\bar{\mathscr{T}}_q^{(k,l)}$ are in fact appropriately normalized versions of the same set of "tau" matrices $\tau_{k,q}^{(n)}$, namely

$$\lambda_q^{(k,l)} \mathscr{T}_q^{(k,l)} = \bar{H}_q^{(k+l)} \tau_{k,q}^{(k+l)}, \qquad \lambda_q^{(k,l)} \bar{\mathscr{T}}_q^{(k,l)} = H_q^{(k+l)} \tau_{-l,r}^{(N-k-l)} \tag{12}$$

The constant factors are obtained by modifying FR to include the twists. Working through Eqs. (FR 3.20–40), it may be shown that the star weight, with $\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j$ restricted to an allowed configuration for $\mathscr{T}_q^{(k,l)}$, is given by

$$\begin{aligned}&U_{(j)qr}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) \\ &\quad = \omega^{-k(\sigma'_j - \sigma'_{j+1} + n_{2j-1} + n_{2j})} (b_q/d_q)^{\sigma_j - \sigma_{j+1} + \sigma'_{j+1} - \sigma'_j} \\ &\quad \times \Omega^{kl}_{p_{2j-1}q} \bar{\Omega}^{kl}_{p_{2j}q} \bar{h}^{k+l}_{p_{2j}q} \sum_{m=0}^{k+l-1} \omega^{m(\sigma'_j - \sigma_{j+1} - k + n_{2j-1} + n_{2j})} \frac{\eta_{q,k+l,\sigma_{j+1} - \sigma'_{j+1} + k}}{\eta_{q,k+l,m}} \\ &\quad \times F_{p_{2j-1}q}(k+l, \sigma_j - \sigma'_j + k, m)\, F_{p_{2j}q}(k+l, \sigma_{j+1} - \sigma'_{j+1} + k, m)\end{aligned} \tag{13}$$

and when restricted to an allowed configuration for $\bar{\mathscr{T}}_q^{(k,l)}$ it is

$$\begin{aligned}&U_{(j)qr}(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) \\ &\quad = \omega^{l(\sigma'_j - \sigma'_{j+1} + n_{2j-1} + n_{2j})} (c_q/a_q)^{\sigma_j - \sigma_{j+1} + \sigma'_{j+1} - \sigma'_j} \\ &\quad \times \Omega^{kl}_{p_{2j-1}q} \bar{\Omega}^{kl}_{p_{2j}q} h^{k+l}_{p_{2j-1}q} \sum_{m=0}^{N-k-l-1} \omega^{m(\sigma'_j - \sigma_{j+1} + l + n_{2j-1} + n_{2j})} \frac{\eta_{r,N-k-l,\sigma_j - \sigma'_j - l}}{\eta_{r,N-k-l,m}} \\ &\quad \times F_{p_{2j-1}r}(N-k-l, \sigma_j - \sigma'_j - l, m)\, F_{p_{2j}r}(N-k-l, \sigma_{j+1} - \sigma'_{j+1} - l, m)\end{aligned} \tag{14}$$

The functions Ω^{kl}_{pq}, $\bar{\Omega}^{kl}_{pq}$ are defined in (FR 3.24), h^j_{pq}, $\bar{h}^j_{pq}$ in (FR 3.35, 36), $\eta_{q,j,m}$ in (FR 3.37), and $F_{pq}(j, \sigma, m)$ in (FR 3.38). The differences introduced by the twist variables are seen in the exponents of ω. Thus, the "first hierarchy" [Eq. (FR 3.46)] is unchanged in form:

$$\lambda_q^{(k,l)} T_q \hat{T}_r = \bar{H}_q^{(k+l)} \tau_{k,q}^{(k+l)} + H_q^{(k+l)} \tau_{-l,r}^{(N-k-l)} \tag{15}$$

as are the constant factors:

$$\lambda_q^{(k,l)} = \prod_{j=1}^{L} (N\Omega^{kl}_{p_{2j-1},q}\bar{\Omega}^{kl}_{p_{2j},q})^{-1}$$
$$H_q^{(n)} = \prod_{j=1}^{L} h^n_{p_{2j-1},q}, \qquad \bar{H}_q^{(n)} = \prod_{j=1}^{L} \bar{h}^n_{p_{2j},q} \tag{16}$$

However, the definition of the "tau" matrices changes to

$$(\tau^{(n)}_{k,q})_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} = \sum_{m_1,\ldots,m_L=0}^{n-1} \prod_{j=1}^{L} S^{k,n}_{(j)q}(m_{j-1}, m_j, \sigma_j, \sigma'_j) \tag{17}$$

where

$$S^{k,n}_{(j)q}(m_{j-1}, m_j, \sigma_j, \sigma'_j)$$
$$= \omega^{-m_{j-1}(\sigma_j - n_{2j-2}) + m_j(\sigma'_j + n_{2j-1}) - k(m_j + n_{2j-2} + n_{2j-1})}$$
$$\times \frac{\eta_{q,n,\sigma_j-\sigma'_j+k}}{\eta_{q,n,m_{j-1}}} F_{p_{2j-2}q}(n, \sigma_j - \sigma'_j + k, m_{j-1}) F_{p_{2j-1}q}(n, \sigma_j - \sigma'_j + k, m_j) \tag{18}$$

$S^{k,n}_{(j)q}(m_{j-1}, m_j, \sigma_j, \sigma'_j)$ is the vertex weight for an n-state by N-state vertex model shown in Fig. 5. This is the viewpoint of Bazhanov and Stoganov.[21] By convention, $\tau^{(0)}_{k,q} = 0$. For $n = 1$, Eq. (FR 3.47b) becomes $\tau^{(1)}_{k,q} = (\prod_{j=1}^{2L} \omega^{n_j} X)^{-k}$, where X is the shift operator (FR 2.42). The shift property (FR 3.51) is modified to

$$\left(\prod_{j=1}^{2L} \omega^{n_j} X\right) \tau^{(n)}_{k,q} = \tau^{(n)}_{k-1,q} \tag{19}$$

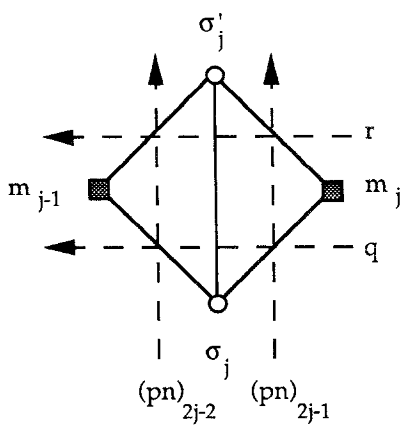

Fig. 5. Vertex weight of Eq. (18).

The arguments leading to (FR 3.50) and (FR 3.51) still hold, that is, we have the commutation relations $[\tau_{k,q}^{(n)}, \tau_{k',q'}^{(n')}]=0$, $[T_q\hat{T}_r, \tau_{k,q}^{(n)}]=0$.

The proof of the "second hierarchy" stems from constructing vectors of the product form $Q(\sigma_1,..., \sigma_L)=g_1(\sigma_1)\cdots g_L(\sigma_L)$ with the property that

$$(\tau_{k,q}^{(2)}Q)(\sigma_1,..., \sigma_L)=Q'(\sigma_1,..., \sigma_L)+Q''(\sigma_1,..., \sigma_L) \tag{20}$$

The method originates in Baxter's papers on the eight-vertex model[22] and depends on the fact that the "tau" matrix is a trace over the "auxiliary variables" m_j. Therefore the left-hand side of (20) may be written as $(\tau_{k,q}^{(2)}Q)_{\boldsymbol{\sigma}}=\mathrm{Tr}[G_1(\sigma_1)\cdots G_L(\sigma_L)]$, where $G_j(\sigma_j)$ is a 2×2 matrix. To recover the right-hand side of (20), one looks for matrices P_j of the form

$$P_j=\begin{pmatrix} 1 & 0 \\ -r_j & 1 \end{pmatrix} \tag{21}$$

such that $P_{j-1}^{-1}G_j(\sigma_j)\,P_j$ is upper triangular, while the functions $g_j(\sigma_j)$ satisfy the periodicity condition $g_j(\sigma_j+N)=g_j(\sigma_j)$. Detailed working is given in Eqs. (FR 4.1)–(FR 4.19): here we simply note modifications needed to accommodate the twists. The most important is that the condition for triangularity, Eq. (FR 4.11b), is changed to

$$\begin{aligned}\frac{g_j(\sigma_j+k)}{g_j(\sigma_j+k-1)}&=\left(\frac{\omega d_{p_{2j-1}}t_q-\omega^{\sigma_j+n_{2j-1}}a_{p_{2j-1}}r_j}{b_{p_{2j-1}}-\omega^{\sigma_j+n_{2j-1}}c_{p_{2j-1}}r_j}\right)\\ &\quad\times\left(\frac{a_{p_{2j-2}}-\omega^{\sigma_j-n_{2j-2}-1}d_{p_{2j-2}}r_{j-1}}{c_{p_{2j-2}}t_q-\omega^{\sigma_j-n_{2j-2}-1}b_{p_{2j-2}}r_{j-1}}\right)\end{aligned} \tag{22}$$

with $t_q=a_qb_q/c_qd_q$. Using this in the periodicity condition gives the same quadratic equation for r_j^N, with the same solution set

$$r_j=(a_q/d_q)\,\omega^{1-\beta_j} \tag{23}$$

Since the integers β_j are arbitrary, there are a total of N^L independent solutions for r_j. The solutions for the $g_j(\sigma_j)$ are modified by the twists in a way which is consistent with (5), that is, on using Eq. (FR 4.19a)

$$\begin{aligned}g_j(\sigma_j+k)&=\frac{\omega^{k(n_{2j-2}+n_{2j-1})}b_{p_{2j-2}}b_{p_{2j-1}}}{c_{p_{2j-2}}c_{p_{2j-1}}}\\ &\quad\times\frac{\bar{W}_{(p)_{2j-2},U_q}(\sigma_j-\beta_{j-1})\,W_{(p)_{2j-1},U_q}(\sigma_j-\beta_j)}{\bar{W}_{(p)_{2j-2},U_q}(0)\,W_{(p)_{2j-1},U_q}(0)}\end{aligned} \tag{24}$$

For each solution of (23) we may calculate $g_j'(\sigma_j) = G_{00}(\sigma_j) - r_j G_{01}(\sigma_j)$ and $g_j''(\sigma_j) = G_{11}(\sigma_j) + r_{j-1} G_{01}(\sigma_j)$ as

$$\begin{aligned} g_j'(\sigma_j) &= \omega^{-k(n_{2j-2}+n_{2j-1})} \left(\frac{a_{p_{2j-2}}}{b_{p_{2j-1}}}\right)\left(1 - \frac{t_q}{t_{p_{2j-2}}}\right) \\ &\quad \times \left(\frac{b_{p_{2j-1}} d_q - \omega^{\sigma_j + n_{2j-1} - \beta_j + 1} c_{p_{2j-1}} a_q}{a_{p_{2j-2}} d_q - \omega^{\sigma_j - n_{2j-2} - \beta_j - 1} d_{p_{2j-2}} a_q}\right) g_j(\sigma_j + k) \\ g_j''(\sigma_j) &= \omega^{-(k-1)(n_{2j-2}+n_{2j-1})} \left(\frac{a_{p_{2j-1}}}{b_{p_{2j-2}}}\right)\left(1 - \frac{\omega t_q}{t_{p_{2j-1}}}\right) \\ &\quad \times \left(\frac{c_{p_{2j-2}} b_q - \omega^{\sigma_j - n_{2j-2} - \beta_j - 1} b_{p_{2j-2}} c_q}{d_{p_{2j-1}} b_q - \omega^{\sigma_j + n_{2j-1} - \beta_j} a_{p_{2j-1}} c_q}\right) g_j(\sigma_j + k) \end{aligned} \tag{25}$$

which is again consistent with the definitions (5). These results give us the extra factors needed to express $g_j(\sigma_j + k)$, $g_j'(\sigma_j)$, $g_j''(\sigma_j)$ in terms of chiral Potts weights using Eqs. (FR 4.19) and therefore to write the modified form of the identity (FR 4.20) as

$$\begin{aligned} &\left(\prod_{j=1}^{L} \frac{y_{p_{2j-2}} y_{p_{2j-1}}}{\xi_{(j), U_q}}\right) \tau_{k,q}^{(2)} \left(\prod_{j=1}^{2L} \omega^{n_j} X\right)^k T_{Uq} \\ &\quad = \left(\prod_{j=1}^{L} \frac{(y_{p_{2j-1}} - \omega x_q)(t_{p_{2j-2}} - t_q)}{(x_{p_{2j-2}} - x_q)\, \xi_{(j),q}}\right) T_q \\ &\quad\quad + \left(\prod_{j=1}^{L} \frac{(y_{p_{2j-2}} - \omega y_q)(t_{p_{2j-1}} - \omega t_q)}{(x_{p_{2j-1}} - y_q)\, \xi_{(j), R^2 q}}\right)\left(\prod_{j=1}^{2L} \omega^{n_j}\right) T_{R^2 q} \end{aligned} \tag{26}$$

where $x_p = a_p/d_p$, $y_p = b_p/c_p$, and $\xi_{(j)q} = \overline{W}_{(p)_{2j-2},q}(0)\, W_{(p)_{2j-1},q}(0)$. Now the remaining arguments of FR go through as before, and the form of the "second hierarchy" of functional equations is

$$\begin{aligned} &\tau_k^{(n)}(t_q)\, \tau_m^{(2)}(\omega^{n-1} t_q) \left(\prod_{j=1}^{2L} \omega^{n_j} X\right)^m \\ &\quad = z(\omega^{n-1} t_q) \left(\prod_{j=1}^{2L} \omega^{n_j}\right) \tau_{k-1}^{(n-1)}(t_q) + \tau_k^{(n+1)}(t_q) \end{aligned} \tag{27}$$

The function $z(t_q)$ is unchanged from Eq. (FR 4.23), namely

$$z(t_q) = \prod_{j=1}^{L} \frac{\omega \mu_{p_{2j-2}} \mu_{p_{2j-1}} (t_{p_{2j-2}} - t_q)(t_{p_{2j-1}} - t_q)}{(y_{p_{2j-2}} y_{p_{2j-1}})^2} \tag{28}$$

with $\mu_p = d_p/c_p$.

3. SUPERINTEGRABILITY WITH TWISTS

A commuting family of transfer matrices generates an infinite sequence of constants of motion. The simplest case is when all vertical rapidities are equal, in which case the Hamiltonian is given in ref. 1. This Hamiltonian comes from the term linear in $q-p$ in the Taylor expansion of T_q about $q=p$. It is expressed in terms of a set of operators X_j, Z_j which act at site j, and satisfy the Z_N commutation relations

$$X_j^N = I, \qquad Z_j^N = I, \qquad Z_j X_j = \omega X_j Z_j \tag{29}$$

A convenient matrix representation is given by

$$(X_j)_{kl} = \delta_{k,l+1} \ (\mathrm{mod}\ N), \quad (Z_j)_{kl} = \delta_{k,l}\omega^k, \quad 0 \leqslant k, l < N \tag{30}$$

Here we will keep the rapidities all equal to p while allowing for arbitrary twists n_j. For the case of no twists, the Hamiltonian may be read off from Eqs. (25) and (26) of ref. 1 as

$$\mathscr{H} = -\sum_{j=1}^{L} \sum_{n=1}^{N-1} (\bar{\alpha}_n X_j^n + \alpha_n Z_j^n Z_{j+1}^{-n}) \tag{31}$$

where

$$\alpha_n = \frac{\exp[i(2n-N)\,\phi/N]}{\sin(\pi n/N)}, \qquad \bar{\alpha}_n = k' \frac{\exp[i(2n-N)\,\bar{\phi}/N]}{\sin(\pi n/N)} \tag{32}$$

and

$$\exp\frac{2i\phi}{N} = \omega^{1/2} \frac{a_p c_p}{b_p d_p}, \qquad \exp\frac{2i\bar{\phi}}{N} = \omega^{1/2} \frac{a_p d_p}{b_p c_p} \tag{33}$$

In order to adapt (31) to our purpose, we need to be careful. The problem is that T_q is not a one-parameter commuting family, rather $T_q \hat{T}_r$ is a two-parameter commuting family. When $q=p$ and $r=Rp$, $T_q \hat{T}_r$ is a constant multiple of the identity matrix: the first term in the expansion about $q=p$, keeping $r=Rp$, gives the Hamiltonian (31). With no twists this distinction is of no consequence.

To utilize (31), we must know how the constants α_n and $\bar{\alpha}_n$ are related to the Taylor expansion of the Boltzmann weights. Write

$$\begin{aligned} W_{pq}(n) &= W_{pp}(0) + (q-p)\, W'_{pp}(n) \\ \bar{W}_{pq}(n) &= \bar{W}_{pp}(0)\, \delta_{n0} + (q-p)\, \bar{W}'_{pp}(n) \end{aligned} \tag{34}$$

in the expansion of $T_q \hat{T}_r$ about $q = p$; then it is clear from (30) that

$$W'_{pp}(n)/W_{pp}(0) = \sum_{k=1}^{N-1} \omega^{kn}\alpha_k, \qquad \bar{W}'_{pp}(n)/\bar{W}_{pp}(0) = \bar{\alpha}_n \tag{35}$$

The Hamiltonian is the sum of two terms, which come from the expansion of $W_{(p)q}(n)$ and $\bar{W}_{(p)q}(n)$, respectively, in the entries of $(T_q \hat{T}_r)_{\sigma\sigma'}$. The term $W_{(p)q}(n)$ gives a diagonal entry $W'_{pp}(\sigma_j - \sigma'_{j+1} + n_{2j-1} + n_{2j})\, \delta_{\sigma_j - \sigma'_{j+1}}$ and $\bar{W}_{(p)q}(n)$ gives the off-diagonal entry $\bar{W}'_{pp}(\sigma_j - \sigma'_j)$. Using (35), this implies the replacement $\alpha_n \to \omega^{n(n_{2j-1} + n_{2j})}\alpha_n$ in (31), while $\bar{\alpha}_n$ remains the same. Consequently, the Hamiltonian becomes

$$\mathscr{H} = -\sum_{j=1}^{L} \sum_{n=1}^{N-1} (\bar{\alpha}_n X_j^n + \alpha_n \omega^{n(n_{2j-1} + n_{2j})} Z_j^n Z_{j+1}^{-n}) \tag{36}$$

Our main interest here is in the "superintegrable" case, which obtains when $\phi = \bar{\phi} = \pi/2$, or alternatively $p = \bar{p}(0, 0)$. Write $\mathscr{H} = \mathscr{H}_0 + k'\mathscr{H}_1$; then we shall show that the two operators $\mathscr{H}_0$, $\mathscr{H}_1$ satisfy the Dolan–Grady conditions

$$\begin{aligned} [\mathscr{H}_0, [\mathscr{H}_0, [\mathscr{H}_0, \mathscr{H}_1]]] &= 4N^2[\mathscr{H}_0, \mathscr{H}_1], \\ [\mathscr{H}_1, [\mathscr{H}_1, [\mathscr{H}_1, \mathscr{H}_0]]] &= 4N^2[\mathscr{H}_1, \mathscr{H}_0] \end{aligned} \tag{37}$$

Note that in ref. 19 there is only one condition, because it is assumed that the two operators are dual. For the chiral Potts model, introducing new variables $\mu_j = \sigma_{j+1} - \sigma_j \pmod N$ is a duality transformation which interchanges operators of the type $\mathscr{H}_0$, $\mathscr{H}_1$, but strict duality only applies when there are no twists. However, if both conditions (37) are satisfied, we do not need strict duality to obtain an Onsager algebra.

We come now to the proof of (37). It is convenient to go to a representation where X_j is diagonal, for then it is shown in ref. 19 that the operator $\mathscr{H}_0$ may be written as

$$\mathscr{H}_0 = \sum_{j=1}^{L} M_j \tag{38}$$

where M_j is the diagonal matrix

$$M_j = \begin{pmatrix} (N-1)/2 & 0 & \cdots & & \\ 0 & (N-3)/2 & 0 & \cdots & \\ \cdots & 0 & \ddots & 0 & \\ & \cdots & 0 & -(N-1)/2 \end{pmatrix} \tag{39}$$

In this representation, $\mathscr{H}_1$ has Z_j replaced by X_j, and we follow ref. 20 by writing

$$X_j^k = P_{k,j} + P^\dagger_{N-k,j} \tag{40}$$

where $P_{k,j}$ is the upper triangular part of X_j^k, and $P^\dagger_{k,j}$ the lower triangular part of X_j^{-k} (raising and lowering operators). The basic commutation relations are

$$[M_j, P_{k,j}] = (N-k)\, P_{k,j}, \qquad [M_j, P^\dagger_{k,j}] = (k-N)\, P^\dagger_{k,j} \tag{41}$$

and these may be used to prove the first of the conditions (37). The second follows from the following duality argument: Use the Z_N commutation relations (29) to write the Dolan–Grady conditions as a set of simultaneous cubic equations in the coefficients α_n, $\bar{\alpha}_n$. For either condition the equations have the same form. In addition, the coefficients α_n, $\bar{\alpha}_n$ are identical in the superintegrable case. Therefore a proof of one of Eqs. (37) provides a proof of both.

As demonstrated in ref. 7, the algebraic structure in the superintegrable case imposes an Ising-like structure on the spectra of associated operators. This structure was used by Baxter to find analytic solutions, in one particular sector, using an inversion identity.[3] We have investigated whether Baxter's technique may be extended to other sectors which may involve twisted boundary conditions. There is in fact one such sector, but numerical calculations indicate that it does not contain the ground state. We report briefly on these investigations here.

First we recall Baxter's arguments.[3] Let $F(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j)$ be the star weight shown in Fig. 6, with q' related to q by an automorphism. We note that this weight is periodic in each of the spin variables. It is shown in ref. 6 that if we choose q' so that

$$a_{q'}, b_{q'}, c_{q'}, d_{q'} = \omega^{-s-1} b_q, \omega^{-s} a_q, \omega^{-s} d_q, c_q \tag{42}$$

where s is any integer, then

$$F(\sigma_j, \sigma_{j+1}, \sigma'_{j+1}, \sigma'_j) = 0 \qquad \text{if } \sigma'_j = \sigma'_{j+1} \text{ and } \sigma'_j \leqslant \sigma_j \leqslant \sigma_{j+1} \leqslant \sigma'_j + N \tag{43}$$

Now introduce vectors $\mathbf{u}_n$, $0 \leqslant n < N$, with entries, in the representation of (30),

$$(\mathbf{u}_n)_\sigma = \delta_{\sigma_1, n} \cdots \delta_{\sigma_L, n} \tag{44}$$

It follows from (43) that the subspace V spanned by the set $\mathbf{u}_n$ is invariant under the operator $T_q \hat{T}_{q'}$. In the representation where X_j are diagonal matrices, there is a natural decomposition into "charge sectors" with quan-

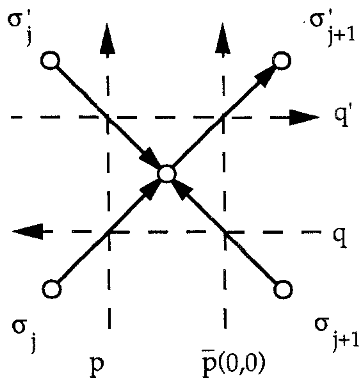

Fig. 6. Star weight for Eq. (43).

tum number $Q=\sum_{j=1}^{L}\sigma_j \pmod N$. In this representation, a natural basis for V contains just one vector $\mathbf{v}_Q$ from each charge sector: therefore $\mathbf{v}_Q$ is an eigenvector of $T_q\hat{T}_{q'}$ for arbitrary q, and we have the functional relation

$$\lambda(q)\,\lambda(q')=g_Q \tag{45}$$

where $\lambda(q)$ is an eigenvalue and g_Q may be computed from $F(0,0,n,n)$. Under the action of the family T_q, $\mathbf{v}_Q$ generates an irreducible eigenvector whose eigenvalues all satisfy (45).[7] This may be solved to obtain all the eigenvalues in that sector.

Now the automorphisms (42) are only a small subset of all possible automorphisms whereby q' may be obtained from q. We have therefore investigated numerically the possibility of obtaining identities similar to (43), and also low-dimensional subspaces which are invariant under $T_q\hat{T}_{q'}$, for all automorphisms generated by the fundamental set R, S, T, U, given in ref. 8. The result is that when the length of the chain plus the twist in the boundary condition is a multiple of N—that is, $L+n_{2j}=0 \pmod N$—there is one other such subspace (and only one). It is spanned by vectors $\mathbf{w}_n$, $0\leqslant n<N$, with components

$$(\mathbf{w}_n)_\sigma=\delta_{\sigma_1,n}\delta_{\sigma_2,n+1}\delta_{\sigma_3,n+2}\cdots\delta_{\sigma_L,n+L-1} \tag{46}$$

Although Baxter's method works in this sector also, it appears to contain the highest state rather than the lowest, and is therefore of little interest.

4. CONCLUSIONS

We have shown that the chiral Potts model may be formulated so that the rapidity lines carry a second integer variable—a 'twist' in the bond crossing it. Here it was introduced explicitly via a change in book-keeping; alternatively, it may be regarded as a use of the automorphism T of Eq. (3). The effect is to change the entire spectrum of the transfer matrix, as, for example, the effect of introducing one antiferromagnetic bond in an Ising model. We have seen that those properties of the chiral Potts model which are essential in the analytic calculation of the eigenvalues remain when twists are introduced. The most important of these properties is the existence of hierarchies of functional equations satisfied by the transfer matrices.

We have also shown that the superintegrable case with twists still satisfies the Dolan–Grady condition, and therefore still has the Onsager algebra structure with resulting Ising-like properties. Unfortunately, there does not appear to be any simple extension of the special inversion identity used so sucessfully by Baxter[3] for the superintegrable model with periodic boundary conditions. However, the algebraic structure should still be of importance in any consideration of the superintegrable chiral Potts model. For example, this structure was used most effectively by Albertini *et al.*[4,5] to make numerical calculations of ground state properties for very long chains, without explicitly diagonalizing the corresponding matrices.

Finally, we mention that the formalism should be useful for considering systems with twisted boundary conditions or with embedded interfaces. Of course, this is of no consequence in calculating bulk properties: however, it should prove to be quite important in investigations related to the theory of conformal invariance for the chiral Potts model, just as in similar investigations using the six-vertex model.[14–18]

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